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# A Relationship between the $1\frac{1}{2}$-Ball Property and the Strong $1\frac{1}{2}$-Ball Property

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The $1\frac{1}{2}$-ball property and the strong $1\frac{1}{2}$-ball property in a Banach space were studied by D. Yost [*Bull. Austral. Math. Soc.* **20** (1979), 285–300; *Math. Scand.* **50** (1982), 100–110]. G. Godini ["Banach Space Theory and Its Applications" (A. Pietsch, N. Popa, and I. Singer, Eds.), Vol. 991, Springer-Verlag, New York/Berlin, 1983], gave geometrical characterizations of the subspaces with property (*), as well as with the $1\frac{1}{2}$-ball property. D. Yost [*Math. Scand.* **50** (1982), 100–110] gave an example that has the $1\frac{1}{2}$-ball property but not the strong $1\frac{1}{2}$-ball property. In the present paper, property (S) is introduced and characterizations of the strong $1\frac{1}{2}$-ball property are given. The subspaces of $C(T)$ which have the $1\frac{1}{2}$-ball property are characterized, where $T$ is compact and connected. 

## 1. INTRODUCTION

In this section, we give some relevant definitions. In Section 2, we study the $1\frac{1}{2}$-ball property. In Section 3, we define the property (S), find a relation between the $1\frac{1}{2}$-ball property and the strong $1\frac{1}{2}$-ball property, and characterize the latter property. In the last section, we show that if $M$ is a finite-dimensional subspace of $C(T)$, where $T$ is a connected compact Hausdorff space, then $M$ has the $1\frac{1}{2}$-ball property if and only if $M$ is the one-dimensional subspace of constant functions.

Let $X$ be a normed linear space, and for each $x \in X$ and $r \geqslant 0$ we denote

$$B(x, r) = \{ y \in X \colon \| y - x \| \leqslant r \}.$$

For a nonempty subset $M$ of $X$ and each $x \in X$ we denote by $P_M(x)$ the set of all best approximations of $x$ from $M$, i.e.,

$$P_M(x) = \{ m_0 \in M \colon \| x - m_0 \| = d(x, M) \}.$$

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We denote $D_M = \{x \in X: P_M(x) \neq \varnothing\}$. The set $M$ is called:

(1) proximinal in $X$ if $D_M = X$.

(2) Chebyshev in $X$ if for each $x \in X$, $P_M(x)$ is a singleton.

Throughout this article, unless otherwise specified, $M$ will denote a linear (not necessarily closed) subspace of $X$. We also denote $P_M^{-1}(0) = \{x \in X: 0 \in P_M(x)\}$.

For $x \in X$ and $\varepsilon > 0$ we denote by $P_M^\varepsilon(x)$ the set of all $\varepsilon$-approximations of $x$ out of $M$, i.e.,

$$P_M^\varepsilon(x) = \{m_0 \in M: \|x - m_0\| \leqslant d(x, M) + \varepsilon\}. \tag{1.1}$$

Notice that for $\varepsilon = 0$, $P_M^0(x) = P_M(x)$. Clearly for each $\varepsilon \geqslant 0$ we have

$$P_M^\varepsilon(x) = M \cap B(x, d(x, M) + \varepsilon) \tag{1.2}$$

and for each $\varepsilon > 0$, $P_M^\varepsilon(x) \neq \varnothing$.

For a set $A \subset X$ and $\varepsilon \geqslant 0$, the closure of the $\varepsilon$-neighborhood of $A$ is the set

$$A_\varepsilon = \overline{B_\varepsilon(A)} = \{x \in X: d(x, A) \leqslant \varepsilon\}.$$

Using the convention that $d(x, \varnothing) = \infty$, it follows that for $A = \varnothing$ we have $A_\varepsilon = \varnothing$ for each $\varepsilon \geqslant 0$.

*Remarks* 1.1 [10]. (1) For each $x \in X$ and $0 \leqslant \varepsilon_1 \leqslant \varepsilon_2$ we have

$$P_M^{\varepsilon_1}(x)_{\varepsilon_2 - \varepsilon_1} \subset B(x, d(x, M) + \varepsilon_2).$$

(2) Let $x \in D_M$. The following statements are equivalent.

(i) $d(m, P_M(x)) = \|x - m\| - d(x, M)$ for any $m \in M$;

(ii) $P_M^\varepsilon(x) = P_M(x)_\varepsilon \cap M$ for any $\varepsilon \geqslant 0$.

LEMMA 1.2. *Let $M$ be a subspace of $X$, $\varepsilon > 0$, and $x \in X$. Then*

(1) $P_M^\varepsilon(x) \neq \varnothing$.

(2) *$P_M^\varepsilon(x)$ is a closed, bounded, convex subset of $M$.*

*Proof.* (1) Since $\varepsilon > 0$, it is clear.

(2) Let $m \in P_M^\varepsilon(x)$. Then $\|x - m\| \leqslant d(x, M) + \varepsilon$. So $\|m\| \leqslant \|x\| + d(x, M) + \varepsilon$. Thus $P_M^\varepsilon(x)$ is bounded.

Let $\{m_n\} \subset P_M^\varepsilon(x)$ satisfy $m_n \to m$. Since $\{m_n\} \subset P_M^\varepsilon(x)$, $\|x - m_n\| \leqslant d(x, M) + \varepsilon$ for each $n$. Taking the limit as $n \to \infty$, $\|x - m\| \leqslant d(x, M) + \varepsilon$. Thus $m \in P_M^\varepsilon(x)$, so $P_M^\varepsilon(x)$ is closed.

Let $m_1, m_2 \in P^{\varepsilon}_M(x)$ and $0 \leqslant \lambda \leqslant 1$. Then

$$\begin{aligned}\|x - \lambda m_1 - (1-\lambda)m_2\| &= \|\lambda(x - m_1) + (1-\lambda)(x - m_2)\| \\ &\leqslant \lambda\, \|x - m_1\| + (1-\lambda)\, \|x - m_2\| \\ &\leqslant d(x, M) + \varepsilon.\end{aligned}$$

Thus $\lambda m_1 + (1-\lambda)m_2 \in P^{\varepsilon}_M(x)$ and $P^{\varepsilon}_M(x)$ is convex.

## 2. The $1\frac{1}{2}$-Ball Property

D. Yost defined and studied the $1\frac{1}{2}$-ball property. G. Godini generalized the concept of semi-$L$-summand—proprty (*). By using it, she gave geometrical characterization of the $1\frac{1}{2}$-ball property.

Definition 2.1 [17]. A subspace $M$ of a normed linear space $X$ has the *$1\frac{1}{2}$-ball property* in $X$ if the conditions $m \in M$, $x \in X$, $r_i \geqslant 0$ $(i = 1, 2)$, $M \cap B(x, r_2) \neq \varnothing$, and $\|x - m\| < r_1 + r_2$ imply that $M \cap B(m, r_1) \cap B(x, r_2) \neq \varnothing$.

Definition 2.2 [14]. Let $M$ be a subspace of a normed linear space $X$. $M$ is called a *semi-$L$-summand* in $X$ if $M$ is Chebyshev in $X$ and the metric projection $P_M: x \to M$ satisfies

$$\|x\| = \|P_M(x)\| + \|x - P_M(x)\|$$

for each $x \in X$.

Definition 2.3 [10]. The subspace $M$ of $X$ is said to have *property* (*) in $X$, if for each $x \in D_M$ and each $m \in M$ we have that

$$d(m, P_M(x)) = \|x - m\| - d(x, M).$$

*Remark* 2.4. Note that when $M$ is Chebyshev with property (*), then for each $x \in X$,

$$\|x - m\| = \|x - P_M(x)\| + \|m - P_M(x)\| \qquad \text{for any } m \in M.$$

Thus when $M$ is Chebyshev, the following statements are equivalent.

(i) $M$ is a semi-$L$-summand;

(ii) $M$ has property (*).

Theorem 2.5 [10]. *Let $M$ be a linear subspace of $X$. The following statements are equivalent.*

(1) $M$ has the $1\frac{1}{2}$-ball property in $X$;

(2) The relations $x \in X$, $r_1, r_2 \geqslant 0$ with $d(x, M) \leqslant r_1 < r_2$, $A_i = \{m \in M: \|x - m\| = r_i\}$ $(i = 1, 2)$, $A_1 \neq \varnothing$, and $m_2 \in A_2$ imply that

$$d(m_2, A_1) = r_2 - r_1;$$

(3) For each $x \in X$ and $0 \leqslant \varepsilon_1 < \varepsilon_2$ we have

$$P_M^{\varepsilon_2}(x) = P_M^{\varepsilon_1}(x)_{\varepsilon_2 - \varepsilon_1} \cap M,$$

whenever $P_M^{\varepsilon_1}(x) \neq \varnothing$.

COROLLARY 2.6 [10]. *Let $M$ be a linear subspace of $X$.*

(1) *If $M$ has the $1\frac{1}{2}$-ball property in $X$, then $M$ has property (*) in $X$.*

(2) *If $M$ is proximinal and has property (*) in $X$, then $M$ has the $1\frac{1}{2}$-ball property in $X$.*

Corollary 2.6 suggests the problem of finding a subspace which has property (*) and the $1\frac{1}{2}$-ball property, but is not proximinal.

EXAMPLE 2.7. [There is a subspace which has property (*) and the $1\frac{1}{2}$-ball property, but is not proximinal]. Let $M$ be a dense proper subspace of a normed linear space $X$. For each $x \in X \backslash M$, $d(x, M) = 0$, but $x \notin M$. Then $D_M = M$; i.e., $M$ is not proximinal. Let $x \in D_M$. Then $d(m, P_M(x)) = \|m - x\|$ and $\|x - m\| - d(x, M) = \|x - m\|$ for each $m \in M$, so $d(m, P_M(x)) + d(x, M) = \|x - m\|$ for each $m \in M$. Thus $M$ has property (*) in $X$. Now we want to show that $M$ has the $1\frac{1}{2}$-ball property in $X$. Let $m \in M$, $x \in X$, $r_i \geqslant 0$, $i = 1, 2$, $M \cap B(x, r_2) \neq \varnothing$, and $\|x - m\| < r_1 + r_2$. Since $\|x - m\| < r_1 + r_2$, $B(m, r_1) \cap B(x, r_2) \neq \varnothing$, so $[x, m] \cap B(m, r_1) \cap B(x, r_2) \neq \varnothing$. There exist $x_0, x_0' \in X$ such that $[x_0, x_0'] = [x, m] \cap B(m, r_1) \cap B(x, r_2)$, $\|x - x_0'\| = r_2$, and $\|m - r_2\| = r_1$. Assume $x_0 = x_0'$. Then $\|x - m\| = \|x - x_0\| + \|x_0 - m\| = r_1 + r_2$. This is a contradiction. Thus $x_0 \neq x_0'$ and $\|x - x_0\| < r_2$ and $\|m - x_0'\| < r_1$.

Claim: $\frac{1}{2}(x_0 + x_0') \in B^0(m, r_1) \cap B^0(x, r_2)$;

$$\|x - \tfrac{1}{2}(x_0 + x_0')\| = \tfrac{1}{2}\|x - x_0\| + \tfrac{1}{2}\|x - x_0'\| < r_2$$

$$\|m - \tfrac{1}{2}(x_0 + x_0')\| = \tfrac{1}{2}\|m - x_0\| + \tfrac{1}{2}\|m - x_0'\| < r_1.$$

Thus $\frac{1}{2}(x_0 + x_0') \in B^0(m, r_1) \cap B^0(x, r_2)$. In particular, $B^0(m, r_1) \cap B^0(x, r_2)$ is a nonempty open set with $\varnothing \neq M \cap B^0(m, r_1) \cap B^0(x, r_2) \subset M \cap B(m, r_1) \cap B(x, r_2)$. Hence $M$ has the $1\frac{1}{2}$-ball property in $X$.

COROLLARY 2.8 [10]. *Let $M$ be a complete subspace of $X$. Then $M$ has* *the $1\frac{1}{2}$-ball property in $X$ if and only if $M$ is proximinal and has property $(*)$ in $X$.*

*Remark* 2.9. D. Yost [17] proved that if $M$ is a closed subspace of a Banach space $X$ which has the $1\frac{1}{2}$-ball property, then $M$ is proximinal in $X$ and $P_M$ is Lipschitz continuous.

COROLLARY 2.10. *Let $M$ be a linear subspace of a Banach space $X$. The following statements are equivalent.*

(1) *$M$ is proximinal and has property $(*)$;*

(2) *For each $x \in X$ and $\varepsilon_1, \varepsilon_2 \geqslant 0$,*

$$P_M^{\varepsilon_1}(x)_{\varepsilon_2} \cap M = P_M^{\varepsilon_2}(x)_{\varepsilon_1} \cap M.$$

*Proof.* (1) $\Rightarrow$ (2) Suppose that (1) holds. Let $x \in X$ and $\varepsilon_1, \varepsilon_2 \geqslant 0$. If $\varepsilon_1 = \varepsilon_2 = 0$, then it is clear. If one of $\varepsilon_1$ and $\varepsilon_2$ is zero, there is nothing to prove from Remarks 1.1. Thus we may assume $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$. Put $\varepsilon = \varepsilon_1 + \varepsilon_2$. Since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, $0 < \varepsilon_1 < \varepsilon$ and $0 < \varepsilon_2 < \varepsilon$. By Theorem 2.5 and Corollary 2.6,

$$P_M^{\varepsilon}(x) = P_M^{\varepsilon_1}(x)_{\varepsilon - \varepsilon_1} \cap M = P_M^{\varepsilon_1}(x)_{\varepsilon_2} \cap M,$$

and

$$P_M^{\varepsilon}(x) = P_M^{\varepsilon_2}(x)_{\varepsilon - \varepsilon_2} \cap M = P_M^{\varepsilon_2}(x)_{\varepsilon_1} \cap M.$$

Thus $P_M^{\varepsilon_1}(x) \cap M = P_M^{\varepsilon_2}(x) \cap M$.

(2) $\Rightarrow$ (1) Suppose that (2) holds. First we will prove that $M$ is proximinal. Suppose not, i.e., there exists $x \in X$ such that $x \notin D_M$. By (2),

$$P_M(x)_{\varepsilon_2} \cap M = P_M^{\varepsilon_2}(x) \cap M \text{ if } \varepsilon_1 = 0 \qquad \text{and} \qquad \varepsilon_2 > 0.$$

Since $P_M(x) = \varnothing$, $P_M(x)_{\varepsilon_2} = \varnothing$ so $P_M(x)_{\varepsilon_2} \cap M = \varnothing$. But $P_M^{\varepsilon_2}(x) \cap M = P_M^{\varepsilon_2}(x) \neq \varnothing$ since $\varepsilon_2 > 0$. This is a contradiction. Thus $M$ is proximinal. Finally we must show that $M$ has property $(*)$. Let $x \in X$. Put $\varepsilon_1 \geqslant 0$ and $\varepsilon_2 = 0$. Then

$$P_M^{\varepsilon_1}(x) = P_M(x)_{\varepsilon_1} \cap M.$$

By Remarks 1.1, $M$ has property $(*)$.

Combining Corollaries 2.8 and 2.10, we obtain the following Corollary.

COROLLARY 2.11. *Let $M$ be a complete subspace of a Banach space $X$. Then the following statements are equivalent.*

(1) *M has the $1\frac{1}{2}$-ball property in X;*

(2) *M is proximinal and has property (*) in X;*

(3) *For each $x \in X$ and $\varepsilon_1, \varepsilon_2 \geqslant 0$,*

$$P_M^{\varepsilon_1}(x)_{\varepsilon_2} \cap M = P_M^{\varepsilon_2}(x)_{\varepsilon_1} \cap M.$$

THEOREM 2.12 [9]. *Let M be a finite-dimensional subspace of X. If $P_M$ is Lipschitz continuous, then $P_M$ has a Lipschitz continuous selection which is homogeneous and additive modulo M.*

COROLLARY 2.13. *If a finite-dimensional subspace M has the $1\frac{1}{2}$-ball property, then $P_M$ has a Lipschitz continuous selection which is homogeneous and additive modulo M.*

*Remark.* D. Yost [17] proved that if $M$ has the $1\frac{1}{2}$-ball property, then $P_M$ has a continuous selection which is homogeneous and additive modulo $M$.

## 3. THE STRONG $1\frac{1}{2}$-BALL PROPERTY

In this section we will define property (S) to characterize the strong $1\frac{1}{2}$-ball property.

DEFINITION 3.1. Let $M$ be a subspace of a normed linear space $X$. We say that $M$ has *property* (S) in $X$ if for each $x \in X$ and each $\varepsilon \geqslant 0$ with $P_M^\varepsilon(x) \neq \emptyset$, $P_M^\varepsilon(x)$ is proximinal in $M$.

LEMMA 3.2. *Let M be a subspace of a normed linear space X, $x \in X$ and $\varepsilon > 0$. Assume that $P_M(x)$ is proximinal in M. If $m_0 \in M \backslash P_M(x)_\varepsilon$, then*

$$d(m_0, P_M(x)_\varepsilon \cap M) = d(m_0, P_M(x)) - \varepsilon.$$

*Proof.* Claim: $d(m_0, P_M(x)) - \varepsilon \leqslant d(m_0, P_M(x)_\varepsilon \cap M)$. Suppose not, i.e., there exists $m^\varepsilon \in P_M(x)_\varepsilon \cap M$ such that $\|m_0 - m^\varepsilon\| < d(m_0, P_M(x)) - \varepsilon$. Then

$$\begin{aligned}
\varepsilon &< d(m_0, P_M(x)) - \|m_0 - m^\varepsilon\| \\
&= \inf_{m \in P_M(x)} \|m_0 - m\| - \|m_0 - m^\varepsilon\| \\
&= \inf_{m \in P_M(x)} \{\|m_0 - m\| - \|m_0 - m^\varepsilon\|\} \\
&\leqslant \inf_{m \in P_M(x)} \|m - m^\varepsilon\| = d(m^\varepsilon, P_M(x)) \leqslant \varepsilon
\end{aligned}$$

since $m^{\varepsilon} \in P_M(x)_{\varepsilon}$. This is a contradiction. Thus $d(m_0, P_M(x)) - \varepsilon \leqslant d(m_0, P_M(x)_{\varepsilon} \cap M)$ and the claim is proved.

Next we want to prove that $d(m_0, P_M(x)) - \varepsilon = d(m_0, P_M(x)_{\varepsilon} \cap M)$. Suppose that $d(m_0, P_M(x)) - \varepsilon < d(m_0, P_M(x)_{\varepsilon} \cap M)$. Since $P_M(x)$ is proximinal in $M$, there exists $m' \in P_M(x)$ such that $\|m_0 - m'\| = d(m_0, P_M(x))$. Then

$$m' + \frac{\varepsilon(m_0 - m')}{\|m_0 - m'\|} \in P_M(x)_{\varepsilon} \cap M$$

and

$$\begin{aligned} d(m_0, P_M(x)) - \varepsilon &= \left\| m_0 - m' - \frac{\varepsilon(m_0 - m')}{\|m_0 - m'\|} \right\| \\ &\geqslant d(m_0, P_M(x)_{\varepsilon} \cap M). \end{aligned}$$

This is a contradiction. Thus $d(m_0, P_M(x)_{\varepsilon} \cap M) = d(m_0, P_M(x)) - \varepsilon$.

*Remark* 3.3. For any proximinal subset $A$ of $X$ and for each $\varepsilon > 0$, $A_{\varepsilon}$ is also proximinal in $X$. We can prove it by a similar argument to Lemma 3.2.

LEMMA 3.4. *Let $M$ be a subspace with property* (*) *in $X$ and $x \in X$. If $P_M(x)$ is proximinal in $M$, then $P^{\varepsilon}_M(x)$ is proximinal in $M$ for each $\varepsilon > 0$.*

*Proof.* Let $\varepsilon > 0$ be given. Suppose that $P_M(x)$ is proximinal in $M$. Let $m_0 \in M$ be fixed. If $m_0 \in P^{\varepsilon}_M(x)$, there is nothing to prove. So we may assume $m_0 \notin P^{\varepsilon}_M(x)$. Then $d(m_0, P^{\varepsilon}_M(x)) > 0$ and $\|x - m_0\| > d(x, M) + \varepsilon$. Since $P_M(x) \neq \varnothing$ is proximinal in $M$, there exists $m' \in M$ such that $\|m_0 - m'\| = d(m_0, P_M(x))$ and $\|x - m'\| = d(x, M)$. Then

$$\begin{aligned} \|m_0 - m'\| &\geqslant \|x - m_0\| - \|x - m'\| \\ &> d(x, M) + \varepsilon - d(x, M) = \varepsilon. \end{aligned}$$

Claim:

$$m' + \frac{\varepsilon(m_0 - m')}{\|m_0 - m'\|} \in P_{P^{\varepsilon}_M(x)}(m_0).$$

Since $m_0, m' \in M$, $m' + \varepsilon(m_0 - m')/\|m_0 - m'\| \in M$. Since

$$\left\| x - m' - \frac{\varepsilon(m_0 - m')}{\|m_0 - m'\|} \right\| \leqslant \|x - m'\| + \varepsilon = d(x, M) + \varepsilon,$$

$$m' + \frac{\varepsilon(m_0 - m')}{\|m_0 - m'\|} \in P^{\varepsilon}_M(x).$$

By Remarks 1.1 and Lemma 3.2,

$$\begin{aligned}\left\| m_0 - m' - \frac{\varepsilon(m_0 - m')}{\|m_0 - m'\|} \right\| &= \|m_0 - m'\| - \varepsilon \\ &= d(m_0, P_M(x)) - \varepsilon \\ &= d(m_0, P_M(x)_\varepsilon \cap M) \\ &= d(m_0, P^\varepsilon_M(x)).\end{aligned}$$

So $m' + \varepsilon(m_0 - m')/\|m_0 - m'\| \in P_{P^\varepsilon_M(x)}(m_0)$. Thus $P^\varepsilon_M(x)$ is proximinal in $M$.

THEOREM 3.5. *Let $M$ be a subspace of $X$ which has property* (*) *in $X$. Then the following statements are equivalent.*

(1) *$M$ has property* (S) *in $X$;*

(2) (i) *for each $x \in D_M$, $P_M(x)$ is proximinal in $M$.*

(ii) *for each $x \in X \backslash D_M$, $P^\varepsilon_M(x)$ is proximinal in $M$ for each $\varepsilon > 0$.*

*Proof.* By the definition of property (S), (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By Lemma 3.4, for each $x \in D_M$, $P^\varepsilon_M(x)$ is proximinal in $M$ for each $\varepsilon > 0$ when $P_M(x)$ is proximinal in $M$. Thus $M$ has property (S) in $X$.

COROLLARY 3.6. *Let $M$ be a proximinal subspace of $X$ which has property* (*) *in $X$. Then the following statements are equivalent.*

(1) *$M$ has property* (S) *in $X$;*

(2) *For each $x \in X$, $P_M(x)$ is proximinal in $M$.*

EXAMPLES 3.7. (1) Every closed subspace of a Hilbert space has property (S).

(2) Every finite-dimensional subspace of any normed linear space has property (S).

Indeed, let $M$ be a finite-dimensional subspace of $X$. Since all sets $P^\varepsilon_M(x)$ are compact, $P^\varepsilon_M(x)$ is proximinal in $M$.

(3) Every Chebyshev subspace with property (*) has property (S).

Indeed, let $M$ be a Chebyshev subspace with property (*). Since for each $x \in X$, $P_M(x)$ is a singleton, $P_M(x)$ is proximinal in $M$. By Corollary 3.6, $M$ has property (S).

(4) Every subspace having the strong $1\frac{1}{2}$-ball property [Definition 3.8] has property (S). [We will prove this later.]

DEFINITION 3.8 [18]. A subspace $M$ of a normed linear space $X$ is said to have the *strong* $1\frac{1}{2}$-*ball property* in $X$ if the conditions $m \in M$, $x \in M$, $r_i \geqslant 0$ $(i = 1, 2)$, $M \cap B(x, r_2) \neq \varnothing$, and $\|x - m\| \leqslant r_1 + r_2$ imply that $M \cap B(m, r_1) \cap B(x, r_2) \neq \varnothing$.

Clearly the strong $1\frac{1}{2}$-ball property implies the $1\frac{1}{2}$-ball property. But the converse is not true as D. Yost [18] has shown.

EXAMPLE 3.9 [18]. (1) Suppose that $M$ has the $1\frac{1}{2}$-ball property in $X$. If $M$ is reflexive, or if $X$ is a dual space and $M$ a weak (*) closed subspace, then $M$ has the strong $1\frac{1}{2}$-ball property.

(2) Let $X$ be the disc algebra (i.e., the sup normed space of functions continuous on $\Delta$, the closed unit in $\mathbb{C}$, and analytic on the interior of $\Delta$). Let $M = \{x \in X: x(1) = 0\}$. Then $M$ has the $1\frac{1}{2}$-ball property but does not have the strong $1\frac{1}{2}$-ball property.

THEOREM 3.10. *Let $M$ be a linear subspace of a normed linear space $X$. If $M$ has the strong $1\frac{1}{2}$-ball property in $X$, then $M$ has property* (S).

*Proof.* Let $x \in X$. If $x \in D_M$, then $M \cap B(x, \delta) \neq \varnothing$, where $\delta = d(x, M)$. Let $m \in M$ be given. If $m \in P_M(x)$, then there is nothing to prove. We may assume $m \notin P_M(x)$. Set $r = d(m, P_M(x))$. Since $M$ has property (*), $d(m, P_M(x)) = \|x - m\| - d(x, M)$. Thus $\|x - m\| = r + \delta$. Since $M$ has the strong $1\frac{1}{2}$-ball property in $X$,

$$M \cap B(m, r) \cap B(x, \delta) \neq \varnothing.$$

Choose $m_0 \in M \cap B(m, r) \cap B(x, \delta)$. Then $\|m - m_0\| \leqslant r = d(m, P_M(x))$ and $m_0 \in P_M(x)$. Thus $\|m - m_0\| = d(m, P_M(x))$. Hence $P_M(x)$ is proximinal in $M$. By Lemma 3.4, $P_M^\varepsilon(x)$ is proximinal in $M$ for any $\varepsilon > 0$. If $x \notin D_M$, then $M \cap B(x, \delta) = \varnothing$ where $\delta = d(x, M)$. So $P_M(x) = \varnothing$, but $P_M^\varepsilon(x) \neq \varnothing$ for $\varepsilon > 0$. We want to prove that $P_M^\varepsilon(x)$ is proximinal in $M$ for $\varepsilon > 0$. Let $m \in M$ and $\varepsilon > 0$ be fixed. If $m \in P_M^\varepsilon(x)$, then there is nothing to prove. If $m \notin P_M^\varepsilon(x)$, then $\|x - m\| > d(x, M) + \varepsilon$. Let $r_1 = d(x, M) + \varepsilon$ and $r_2 = \|x - m\|$. Since $P_M^\varepsilon(x) \neq \varnothing$, there exists $m_1 \in P_M^\varepsilon(x)$ such that $\|x - m_1\| \leqslant d(x, M) + \varepsilon = r_1$. Since $\|x - m\| > d(x, M) + \varepsilon = r_1$, there exists $m' \in [m_1, m]$ such that $\|x - m'\| = r_1$, where $[m_1, m] = \{\lambda m_1 + (1 - \lambda)m: 0 \leqslant \lambda \leqslant 1\}$. Therefore

$$A_1 = \{m_0 \in M: \|x - m_0\| = r_1\} \neq \varnothing$$

and

$$m \in A_2 = \{m_0 \in M: \|x - m_0\| = r_2\}.$$

By Theorem 2.5,

$$d(m, A_1) = r_2 - r_1 = \|x - m\| - d(x \cdot M) - \varepsilon.$$

Claim: $d(m, A_1) = d(m, P^{\varepsilon}_M(x))$. Clearly $d(m, P^{\varepsilon}_M(x)) \leqslant d(m, A_1)$ since $A_1 \subset P^{\varepsilon}_M(x)$. Suppose $d(m, P^{\varepsilon}_M(x)) < d(m, A_1)$. Then there exists $m_0 \in P^{\varepsilon}_M(x) \backslash A_1$ such that $\|m - m_0\| < d(m, A_1)$. Since $\|x - m\| > d(x, M) + \varepsilon$ and $\|x - m_0\| < d(x, M) + \varepsilon$, there exists $m_0^* \in [m, m_0]$ such that $\|x - m_0^*\| = d(x, M) + \varepsilon$, i.e., $m_0^* \in A_1$ and $\|m - m_0^*\| < \|m - m_0\|$. This is a contradiction to $\|m - m_0\| < d(m, A_1)$. Thus $d(m, P^{\varepsilon}_M(x)) = d(m, A_1) = r_2 - r_1$. Since $M \cap B(x, r_1) \neq \varnothing$ and $\|x - m\| = r_2 = (r_2 - r_1) + r_1$, $M \cap B(m, r_2 - r_1) \cap B(x, r_1) \neq \varnothing$. Choose $m_0^{**} \in M \cap B(m, r_2 - r_1) \cap B(x, r_1)$. Then $m_0^{**} \in P^{\varepsilon}_M(x)$ and $\|m - m_0^{**}\| \leqslant r_2 - r_1 = d(m, P^{\varepsilon}_M(x))$. So $\|m - m_0^{**}\| = d(m, P^{\varepsilon}_M(x))$. Thus $P^{\varepsilon}_M(x)$ is proximinal in $M$.

Now we can characterize the strong $1\frac{1}{2}$-ball property.

THEOREM 3.11. *Let $M$ be a subspace of $X$. The following statements are equivalent.*

(1) *$M$ has the strong $1\frac{1}{2}$-ball property in $X$;*

(2) *$M$ has the $1\frac{1}{2}$-ball property and property* (S) *in $X$.*

*Proof.* (1) $\Rightarrow$ (2) Since the strong $1\frac{1}{2}$-ball property implies the $1\frac{1}{2}$-ball property, (1) $\Rightarrow$ (2) follows from Theorem 3.10.

(2) $\Rightarrow$ (1) Since $M$ has the $1\frac{1}{2}$-ball property in $X$, Theorem 2.5 implies that for each $x \in X$ and $0 \leqslant \varepsilon_1 < \varepsilon_2$, we have

$$P^{\varepsilon_2}_M(x) = P^{\varepsilon_1}_M(x)_{\varepsilon_2 - \varepsilon_1} \cap M \tag{3.1}$$

whenever $P^{\varepsilon_1}_M(x) \neq \varnothing$. Let $x \in X$, $m \in M$, $r_1, r_2 \geqslant 0$ be chosen such that $\|x - m\| \leqslant r_1 + r_2$ and $M \cap B(x, r_2) \neq \varnothing$. Then $d(x, M) \leqslant r_2$. If $\|x - m\| \leqslant r_2$, then $m \in M \cap B(m, r_1) \cap B(x, r_2)$. If $\|x - m\| > r_2$, let $\varepsilon_1 = r_2 - d(x, M)$ and $\varepsilon_2 = \|x - m\| - d(x, M)$. Then $0 \leqslant \varepsilon_1 < \varepsilon_2$, $m \in P^{\varepsilon_2}_M(x)$, and $P^{\varepsilon_1}_M(x) = M \cap B(x, r_2) \neq \varnothing$, since $r_2 = d(x, M) + \varepsilon_1$. By (3.1) and $\|x - m\| \leqslant r_1 + r_2$,

$$d(m, P^{\varepsilon_1}_M(x)) \leqslant \varepsilon_2 - \varepsilon_1 = \|x - m\| - r_2 \leqslant r_1.$$

Since $M$ has property (S), $P^{\varepsilon_1}_M(x)$ is proximinal in $M$. Then there exists $m_1 \in P^{\varepsilon_1}_M(x)$ such that $\|m - m_1\| \leqslant r_1$. Since $m_1 \in P^{\varepsilon_1}_M(x)$, $\|x - m_1\| \leqslant d(x, M) + \varepsilon_1 = r_2$. Thus $m_1 \in M \cap B(m, r_1) \cap B(x, r_2)$. Therefore $M$ has the strong $1\frac{1}{2}$-ball property in $X$.

*Remark* 3.12. By Example 3.7 and Theorem 3.10, every Chebyshev subspace with property (*) has the strong $1\frac{1}{2}$-ball property; i.e., every semi-$L$-summand has the strong $1\frac{1}{2}$-ball property.

THEOREM 3.13. *Let $M$ be a closed subspace of a Banach space $X$. The following statements are equivalent.*

(1) *$M$ has the strong $1\frac{1}{2}$-ball property in $X$;*

(2) *$M$ is proximinal with property (*) and property (S) in $X$;*

(3) *$M$ is priximinal with property (*) and for each $x \in X$, $P_M(x)$ is proximinal in $M$;*

(4) *For each $x \in X \backslash M$ there exists $m_x \in P_M(x)$ such that*

$$\|x - m_x\| = \|x\| - \|m_x\|.$$

*Proof.* The equivalence $(1) \Leftrightarrow (2)$ follows from Corollary 2.6 and Theorem 3.11.

$(2) \Leftrightarrow (3)$ The implication $(2) \Rightarrow (3)$ is clear while $(3) \Rightarrow (2)$ follows from Lemma 3.4.

$(3) \Rightarrow (4)$ Suppose that (3) holds. Since $M$ is proximinal with property (*), for each $x \in X$,

$$\|x - m\| = d(x, M) + d(m, P_M(x))$$

for each $m \in M$. Let $x \in X \backslash M$ be given. Since $0 \in M$, $\|x\| = d(x, M) + d(0, P_M(x))$. Since $P_M(x)$ is proximinal in $M$, there exists $m_x \in P_M(x)$ such that $\|x\| = \|x - m_x\| + \|m_x\|$. Thus (4) holds.

$(4) \Rightarrow (3)$ Suppose that (4) holds. Clearly $M$ is proximinal. Let $x \in X \backslash M$ be fixed and $m \in M$. Then $x - m \in X \backslash M$. By (4), there exists $m_{x-m} \in P_M(x-m)$ such that $\|x - m - m_{x-m}\| = \|x - m\| - \|m_{x-m}\|$. Since $m_{x-m} \in P_M(x-m)$, there exists $m' \in P_M(x)$ such that $m_{x-m} = m' - m$. Since $\|m_{x-m}\| = \|m' - m\| = \|x - m\| - \|x - m'\| = \|x - m\| - \|x - m''\| \leqslant \|m - m''\|$ for each $m'' \in P_M(x)$,

$$\|m_{x-m}\| = \|m - m'\| = d(m, P_M(x)).$$

Since $\|x - m - m_{x-m}\| = \|x - m'\| = d(x, M)$, $\|x - m\| = d(x, M) + d(m, P_M(x))$. Since $x \in X$ and $m \in M$ were arbitrary, $M$ has property (*) and for each $x \in X$, $P_M(x)$ is proximinal in $M$. Thus (3) holds.

COROLLARY 3.14. *Let $M$ be a finite-dimensional subspace of a Banach space $X$. The following statements are equivalent.*

(1) *$M$ has the $1\frac{1}{2}$-ball property in $X$;*

(2) *For each* $x \in X \backslash M$ *there exists* $m_x \in P_M(x)$ *such that*

$$\|x - m_x\| = \|x\| - \|m_x\|.$$

*Proof.* Since $M$ is finite dimensional, $M$ has the strong $1\frac{1}{2}$-ball property if and only if it has the $1\frac{1}{2}$-ball property. Thus $(1) \Leftrightarrow (2)$ follows from Theorem 3.13.

*Remark* 3.15. No nontrivial proper subspace in a strictly convex Banach space has the $1\frac{1}{2}$-ball property.

*Proof.* Let $M$ be a subspace of a strictly convex Banach space which has the $1\frac{1}{2}$-ball property. Then, by Remarks 2.9 and 3.12, $M$ has the strong $1\frac{1}{2}$-ball property. By Theorem 3.13, for each $x \in X \backslash M$, there exists $m_x \in P_M(x)$ such that

$$\|x - m_x\| = \|x\| - \|m_x\|.$$

Since $X$ is strictly convex, $x = \alpha m_x$ for some scalar $\alpha$. This is a contradiction to $x \in X \backslash M$.

## 4. THE (STRONG) $1\frac{1}{2}$-BALL PROPERTY IN $C(T)$ WHERE $T$ IS A CONNECTED COMPACT HAUSFORFF SPACE

Let $T$ be a compact Hausdorff space. Then $C(T)$ is the Banach space of real continuous functions defined on $T$ with sup norm:

$$\|f\| = \sup_{t \in T} |f(t)|.$$

If $f \in C(T)$, denote $f^{-1}(0)$ by $Z(f)$ and if $A \subset C(T)$, let $Z(A) = \bigcap \{Z(f) : f \in A\}$.

It is known [4] that a function $f$ of norm one is in $P_M^{-1}(0)$ if and only if there is a continuous linear functional $L$ defined on $C(T)$ such that $L(m) = 0$ for all $m \in M$ and $\|L\| = 1 = L(f)$. In the following lemma, let $f$ be in $P_{M(0)}^{-1}$ with $\|f\| = 1$ and let $L$ be a continuous linear functional on $C(T)$ such that $L(m) = 0$ for all $m$ in $M$ and $\|L\| = 1 = L(f)$.

LEMMA 4.1 [4]. *If* $m$ *is in* $P_M(f)$, *then* $m$ *vanishes on* supp$(L)$.

*Remark.* In the above Lemma, supp$(L)$ is the support of a corresponding regular Borel measure.

LEMMA 4.2 [2, 11]. *Let* $T$ *be a compact Hausdorff space. If* $M$ *is a finite-dimensional subspace of* $C(T)$, *then the following statements are equivalent.*

(1) $P_M$ *is* lsc;

(2) $P_M$ *has a continuous selection* $s$ *with the nulleigenshaft*; *i.e.*, $s(x)=0$ *for each* $x \in P_M^{-1}(0)$;

(3) $Z(P_M(f))$ *is open for each* $f \in P_M^{-1}(0)$.

*Remark.* H. Kruger [11] proved (1)⇔(2). Blatter *et al.* [2] established (1)⇔(3).

THEOREM 4.3. *Let* $T$ *be a connected compact Hausdorff space and* $M$ *a proximinal subspace of* $C(T)$. *If* $Z(P_M(f))$ *is open for each* $f \in P_M^{-1}(0)$, *then* $M$ *is Chebyshev.*

*Proof.* By Lemma 4.2, $Z(P_M(f)) \neq \emptyset$ for any $f \in P_M^{-1}(0)$. Since $T$ is connected and $Z(P_M(f)) \neq \emptyset$, $Z(P_M(f))=T$. Thus $P_M(f)=\{0\}$ for each $f \in \ker P_M$. Hence $M$ is Chebyshev.

COROLLARY 4.4. *Let* $T$ *be a connected compact Hausdorff space and* $M$ *be an n-dimensional subspace of* $C(T)$. *If* $M$ *has the* $1\frac{1}{2}$*-ball property in* $C(T)$, *then* $M$ *is Chebyshev.*

*Proof.* Since $M$ has the $1\frac{1}{2}$-ball property property, $P_M$ is Lipschitz continuous. By Lemma 4.2 and Theorem 4.3, $M$ is Chebyshev.

A. Lima [14] studied the intersections of balls. He defined semi-$L$-summand and gave a characterization of the subspaces in $C(T)$ which are semi-$L$-summands (cf. Definition 2.2).

THEOREM 4.5 [14]. *Let* $M$ *be a closed subspace of* $C(T)$ *where* $T$ *is a compact Hausdorff space. Then* $M$ *is a semi-L-summand in* $C(T)$ *if and only if* $M=C(T)$, $M=\{0\}$, *or* $M=\operatorname{span}(f)$ *for some* $f \in C(T)$ *with* $|f|=1$.

COROLLARY 4.6. *Let* $M$ *be an n-dimensional subspace where* $T$ *is a connected compact Hausdorff space,* $1 \leqslant n < \infty$. *Then the following statements are equivalent*:

(1) $M$ *has the* $1\frac{1}{2}$*-ball property in* $C(T)$;

(2) $M$ *is a semi-L-summand in* $C(T)$;

(3) $M=\operatorname{span}(1)$ *where* $1(t)=1$ *for any* $t \in T$.

*Proof.* (1)⇔(2) By Corollary 2.8, Corollary 4.4, and Definition 2.2, $M$ has the $1\frac{1}{2}$-ball property in $C(T) \Leftrightarrow M$ has property (*) in $C(T)$ and is Chebyshev ⇔ $M$ is a semi-$L$-summand in $C(T)$.

(2)⇔(3) follows from Theorem 4.5.

*Remarks* 4.7. (1) When $T$ is connected and $M$ is finite dimensional in

$C(T)$, semi-$L$-summand, $1\frac{1}{2}$-ball property, and strong $1\frac{1}{2}$-ball property are equivalent properties for $M$.

(2) Let $T$ be a compact Hausdorff space. Assume that $M$ is Chebyshev. Then $M$ has the $1\frac{1}{2}$-ball property in $C(T)$ if and only if $M = C(T)$, $M = (0)$, or $M = \text{span}(f)$ for some $f \in C(T)$ with $|f| = 1$.

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